Warsaw University of Technology
Faculty of Power and Aeronautical Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Institute of Aeronautics and Applied Mechanics

Finite element method 2 (FEM 2)

Introduction to structural dynamics

10.2021

INTRODUCTION TO STRUCTURAL DYNAMICS

The transient dynamic equilibrium equation for a discrete FE model:

$$\underbrace{[M]}_{NDOF \times NDOF} \cdot \underbrace{\{\ddot{q}\}}_{NDOF \times 1} + \underbrace{[C]}_{NDOF \times NDOF} \cdot \underbrace{\{\dot{q}\}}_{NDOF \times 1} + \underbrace{[K]}_{NDOF \times NDOF} \cdot \underbrace{\{q(t)\}}_{NDOF \times 1} = \underbrace{\{F(t)\}}_{NDOF \times 1}$$

- $[M]$: MASS MATRIX
- $\{\ddot{q}\}$: VECTOR OF NODAL ACCELERATIONS
- $[C]$: DAMPING MATRIX
- $\{\dot{q}\}$: VECTOR OF NODAL VELOCITIES
- $[K]$: STIFFNESS MATRIX
- $\{q(t)\}$: VECTOR OF NODAL DISPLACEMENTS
- $\{F(t)\}$: LOAD VECTOR

$$\underset{NDOF\times 1}{\{\dot{q}\}} = \frac{d}{dt}\underset{NDOF\times 1}{\{q(t)\}} = \begin{Bmatrix} \dfrac{dq_1}{dt} \\ \dfrac{dq_2}{dt} \\ \vdots \\ \dfrac{dq_{NDOF}}{dt} \end{Bmatrix}$$

$$\underset{NDOF\times 1}{\{\ddot{q}\}} = \frac{d}{dt}\underset{NDOF\times 1}{\{\dot{q}\}} = \begin{Bmatrix} \dfrac{d^2q_1}{dt^2} \\ \dfrac{d^2q_2}{dt^2} \\ \vdots \\ \dfrac{d^2q_{NDOF}}{dt^2} \end{Bmatrix}$$

SPECIAL CASES :

* HARMONIC VIBRATIONS :

$$\underset{NDOF\times 1}{\{F(t)\}} = \begin{Bmatrix} F_{a_1} \cdot \sin(\omega t + \varphi_1) \\ F_{a_2} \cdot \sin(\omega t + \varphi_2) \\ \vdots \\ \underbrace{F_{a_{NDOF}}}_{\text{load magnitudes}} \cdot \sin(\omega t + \underbrace{\varphi_{NDOF}}_{\text{phase shifts}}) \end{Bmatrix}$$

ω - angular frequency , $\omega = 2\pi \cdot f$ $\left[\frac{1}{s}\right]$

f - frequency $\left[\frac{1}{s}\right]$

* FREE VIBRATIONS :

$$\underset{NDOF\times 1}{\{F(t)\}} = \underset{NDOF\times 1}{\{0\}}$$

* FREE UNDAMPED VIBRATIONS (NATURAL VIBRATIONS) :

$$\underset{NDOF\times 1}{\{F(t)\}} = \underset{NDOF\times 1}{\{0\}} \quad \text{AND} \quad \underset{NDOF\times NDOF}{[C]} = \underset{NDOF\times NDOF}{[0]}$$

* STATIC ANALYSIS :

$$\underset{NDOF\times NDOF}{[M]} = \underset{NDOF\times NDOF}{[0]} \quad , \quad \underset{NDOF\times NDOF}{[C]} = \underset{NDOF\times NDOF}{[0]}$$

$$\underset{NDOF\times 1}{\{F(t)\}} = \underset{NDOF\times 1}{\{F\}} \quad ; \quad F_1, F_2, \ldots = \text{const}$$

NATURAL VIBRATIONS

$$\underset{NDOF\times NDOF}{[M]}\cdot\underset{NDOF\times 1}{\{\ddot{q}\}}+\underset{NDOF\times NDOF}{[K]}\cdot\underset{NDOF\times 1}{\{q(t)\}}=\underset{NDOF\times 1}{\{0\}}$$

general solution :

$$\underset{NDOF\times 1}{\{q(t)\}}=\underset{NDOF\times 1}{\{q\}_A}\cdot\cos\omega t+\underset{NDOF\times 1}{\{q\}_B}\cdot\sin\omega t$$

depend on initial conditions

VELOCITY VECTOR :

$$\underset{NDOF\times 1}{\{\dot{q}(t)\}} = -\omega \underset{NDOF\times 1}{\{q\}_A} \cdot \sin\omega t + \omega \underset{NDOF\times 1}{\{q\}_B} \cdot \cos\omega t$$

ACCELERATION VECTOR :

$$\underset{NDOF\times 1}{\{\ddot{q}(t)\}} = -\omega^2 \underset{NDOF\times 1}{\{q\}_A} \cdot \cos\omega t - \omega^2 \underset{NDOF\times 1}{\{q\}_B} \sin\omega t =$$

$$= -\omega^2 \underset{NDOF\times 1}{\{q(t)\}}$$

$$\underset{NDOF\times NDOF}{[M]}\cdot\left(-\omega^2\underset{NDOF\times 1}{\{q(t)\}}\right)+\underset{NDOF\times NDOF}{[K]}\cdot\underset{NDOF\times 1}{\{q(t)\}}=\underset{NDOF\times 1}{\{0\}}$$

eigenvalue problem

$$\left(\underset{NDOF\times NDOF}{[K]}-\omega^2\underset{NDOF\times NDOF}{[M]}\right)\{q(t)\}=\{0\}$$

nontrivial solution :

- an eigenvector $\{q(t)\}\neq\{0\}$
- $\det\left(\underset{NDOF\times NDOF}{[K]}-\omega^2\underset{NDOF\times NDOF}{[M]}\right)=0$

COMMENT 1:

if $\det([K] - \omega^2 [M]) \neq 0$ then

the inverse matrix $([K]-\omega^2[M])^{-1}$ exists:

$$\underbrace{([K]-\omega^2[M])^{-1} \cdot ([K]-\omega^2[M])}_{=[I]\text{ - identity matrix}} \cdot \{q(t)\} = \underbrace{([K]-\omega^2[M])^{-1} \cdot \{0\}}_{=\{0\}}$$

thus:

$$[I] \cdot \{q(t)\} = \{q(t)\} = \{0\}$$

what is in contradiction to $\{q(t)\} \neq \{0\}$

COMMENT 2: IF $\det[M] \neq 0 \Rightarrow$

$$[M]^{-1} \cdot \left([K] - \omega^2 [M]\right) \cdot \{q(t)\} = [M]^{-1} \cdot \{0\}$$

$$\left(\underbrace{[M]^{-1}[K]}_{[A]} - \omega^2 [I]\right)\{q(t)\} = \{0\}$$

$$[A] \quad , \quad \omega^2 = c \quad , \quad \{q(t)\} = \{v\}$$

$$[A] \cdot \{v\} = c \cdot \{v\}$$

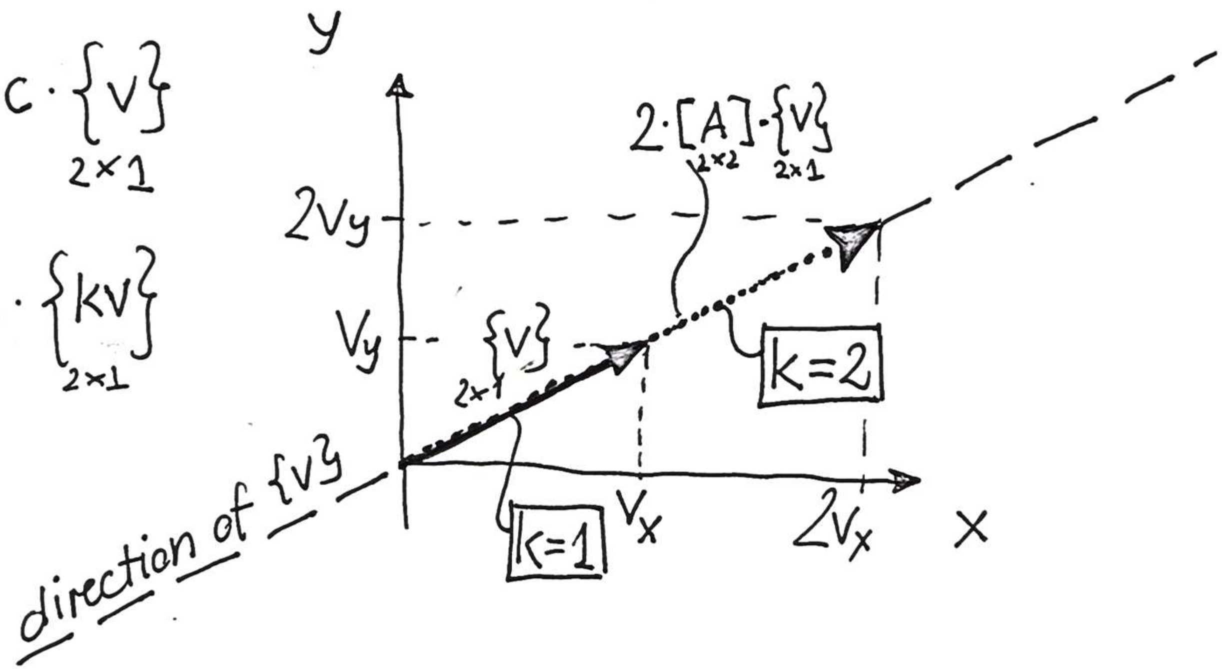
eigenvalue of [A]

eigenvector of [A] corresponding to the eigenvalue c

FOR ANY CONSTANT k (NDOF = 2) :

$$\underset{2\times2}{k\cdot[A]}\cdot\underset{2\times1}{\{v\}} = k\cdot c\cdot\underset{2\times1}{\{v\}}$$

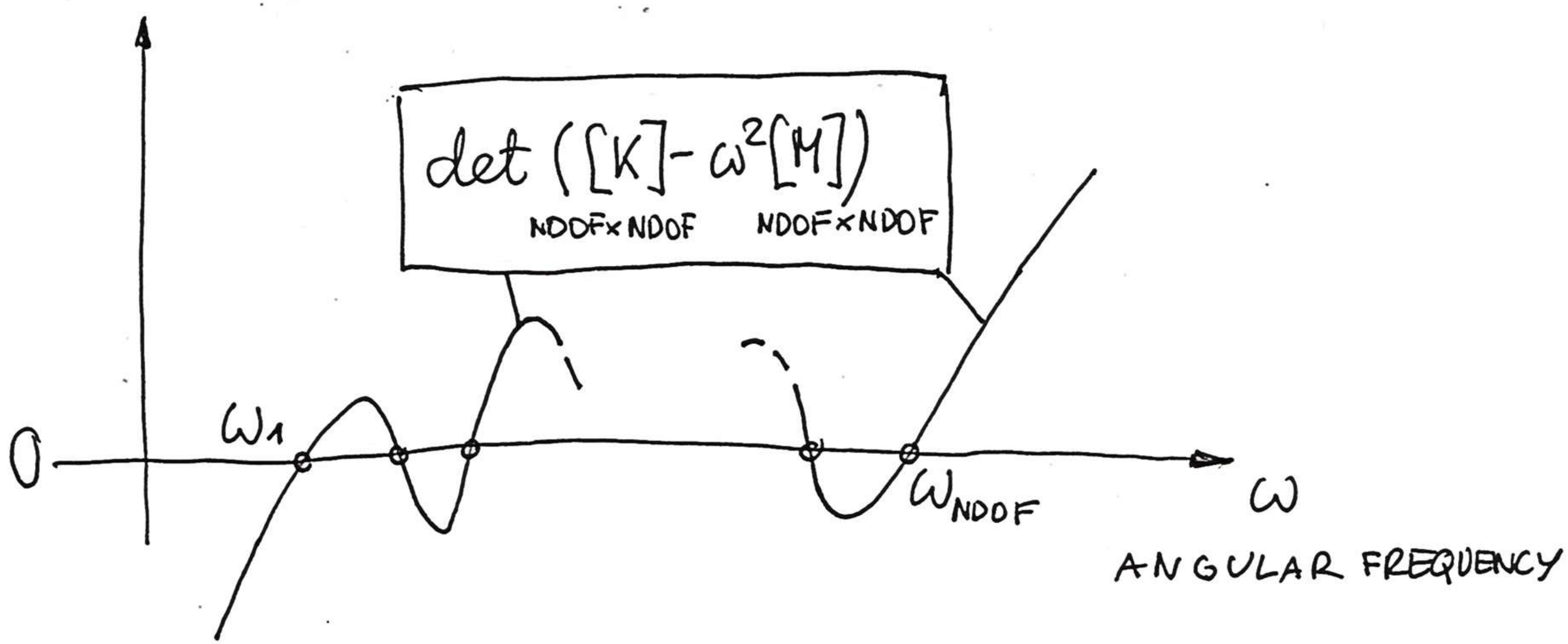
$$\underset{2\times2}{[A]}\cdot\underset{2\times1}{\{kv\}} = c\cdot\underset{2\times1}{\{kv\}}$$



Conclusions :

1) $\dfrac{V_y}{V_x} = \text{const} \Rightarrow V_x$ and V_y are dependent on each other

2) $\{kv\}$ is also the solution of an eigenvalue problem



$\omega_1, \omega_2, \ldots, \omega_i, \ldots \omega_{NDOF}$ – NATURAL ANGULAR FREQUENCIES

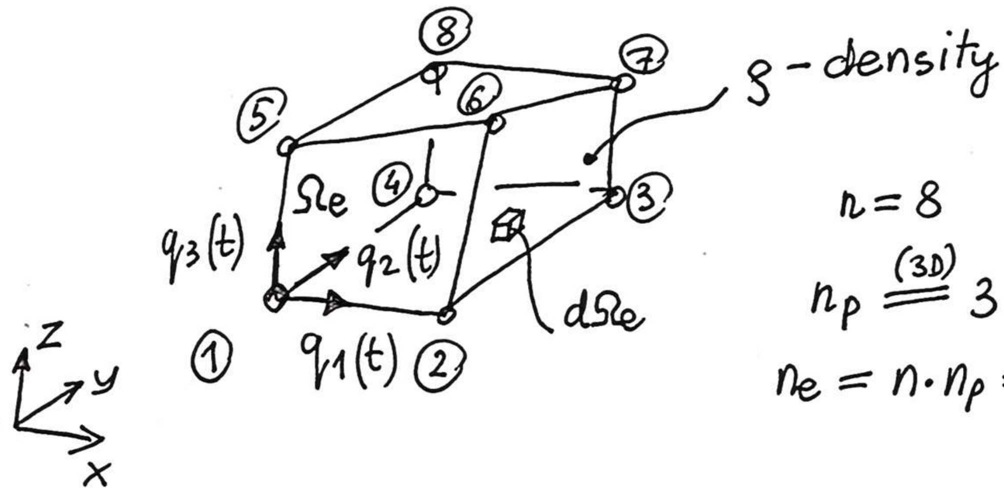
The shape of vibration (the vibration mode) for ω_i is described by the corresponding eigenvector $\{q\}_i$, which is normalized:

$$\underset{1\times NDOF}{\lfloor q \rfloor_i} \cdot \underset{NDOF\times NDOF}{[I]} \underset{NDOF\times 1}{\{q\}_i} = 1 \quad \text{or} \quad \underset{1\times NDOF}{\lfloor q \rfloor_i} \cdot \underset{NDOF\times NDOF}{[M]} \cdot \underset{NDOF\times 1}{\{q\}_i} = 1$$

ANALYSIS OF NATURAL VIBRATIONS (MODAL ANALYSIS)

- is more time-consuming than a linear structural static analysis,
- iterative solver is used for the limited number of ω_i,
- the effect of stress can be included (prestress).

MASS MATRIX OF A FINITE ELEMENT



$n = 8$

$n_p \overset{(3D)}{=\!=} 3$

$n_e = n \cdot n_p = 24$

LOCAL VECTOR OF NODAL PARAMETERS:

$$\underset{1 \times n_e}{\lfloor q(t) \rfloor_e} = \lfloor q_1(t),\ q_2(t),\ \dots,\ q_{n_e}(t) \rfloor_e$$

DISPLACEMENT VECTOR:

$$\underset{3\times 1}{\{u(t)\}} = \begin{Bmatrix} u(t) \\ v(t) \\ w(t) \end{Bmatrix} = \underset{3 \times n_e}{[N]} \cdot \underset{n_e \times 1}{\{q(t)\}_e}$$

VELOCITY VECTOR:

$$\underset{3\times1}{\{\dot{u}\}} = \underset{3\times n_e}{[N]} \cdot \underset{n_e\times1}{\{\dot{q}\}_e} \quad , \quad \underset{1\times3}{\lfloor\dot{u}\rfloor} = \underset{1\times n_e}{\lfloor\dot{q}\rfloor_e} \cdot \underset{n_e\times3}{[N]^T}$$

KINETIC ENERGY OF AN INFINITESIMALLY SMALL PART $d\Omega_e$:

$$dT_e = \frac{1}{2}\underset{1\times3}{\lfloor\dot{u}\rfloor}\, dm \cdot \underset{3\times1}{\{\dot{u}\}} = \frac{1}{2}\underset{1\times n_e}{\lfloor\dot{q}\rfloor_e}\underset{n_e\times3}{[N]^T}\rho\, d\Omega_e \underset{3\times n_e}{[N]}\underset{n_e\times1}{\{\dot{q}\}_e} =$$

$$= \frac{1}{2}\underset{1\times n_e}{\lfloor\dot{q}\rfloor_e} \cdot \rho \underset{n_e\times3}{[N]^T} \cdot \underset{3\times n_e}{[N]} \cdot \underset{n_e\times1}{\{\dot{q}\}_e}\, d\Omega_e$$

KINETIC ENERGY OF A FINITE ELEMENT:

$$T_e = \int dT_e = \int_{\Omega_e} \frac{1}{2} \underset{1\times n_e}{\lfloor \dot{q} \rfloor_e} \cdot \rho \underset{n_e\times 3}{[N]^T} \cdot \underset{3\times n_e}{[N]} \cdot \{\dot{q}\}_e \, d\Omega_e =$$

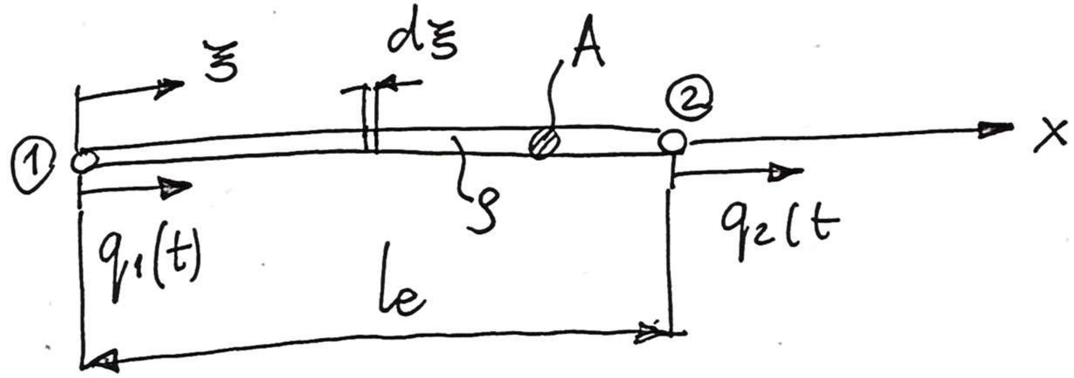
$$= \frac{1}{2} \underset{1\times n_e}{\lfloor \dot{q} \rfloor_e} \cdot \underbrace{\int_{\Omega_e} \rho \underset{n_e\times 3}{[N]^T} \cdot \underset{3\times n_e}{[N]} \, d\Omega_e}_{\underset{n_e\times n_e}{[m]_e}} \cdot \{\dot{q}\}_e =$$

$\underset{n_e\times n_e}{[m]_e}$ — consistent mass matrix

$$= \frac{1}{2} \underset{1\times n_e}{\lfloor \dot{q} \rfloor_e} \cdot \underset{n_e\times n_e}{[m]_e} \cdot \underset{n_e\times 1}{\{\dot{q}\}_e}$$

A lumped mass matrix $[m_L]_e = \frac{\rho \Omega_e}{n_e} \cdot \underset{n_e\times n_e}{[I]}$

MASS MATRIX OF A BAR ELEMENT



$$d\Omega_e = A \cdot d\xi \quad , \quad dm = \varrho \, d\Omega_e = \underbrace{\varrho A}_{\text{linear density } \left[\frac{kg}{m}\right]} \cdot d\xi$$

$$u(\xi, t) = \lfloor N_1(\xi), N_2(\xi) \rfloor \cdot \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}_e$$

$$N_1(\xi) = 1 - \frac{\xi}{l_e} \quad , \quad N_2(\xi) = \frac{\xi}{l_e}$$

$$\dot{u}(\xi) = \lfloor N_1, N_2 \rfloor \cdot \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix}_e = \lfloor \dot{q}_1, \dot{q}_2 \rfloor \cdot \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}$$

$$dT_e = \frac{1}{2}\dot{u}(\xi)\cdot dm \cdot \dot{u}(\xi) = \frac{1}{2}\underset{1\times 2}{\lfloor \dot{q} \rfloor_e} \cdot \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \rho A \cdot \lfloor N_1, N_2 \rfloor \cdot \{\dot{q}\}_e \, d\xi$$

$$T_e = \int dT_e = \frac{1}{2}\underset{1\times 2}{\lfloor \dot{q} \rfloor_e} \cdot \rho A \cdot \underbrace{\int_0^{l_e} \begin{bmatrix} N_1 N_1 & N_1 N_2 \\ N_2 N_1 & N_2 N_2 \end{bmatrix} d\xi}_{\underset{2\times 2}{[m]_e}} \cdot \underset{2\times 1}{\{\dot{q}\}_e}$$

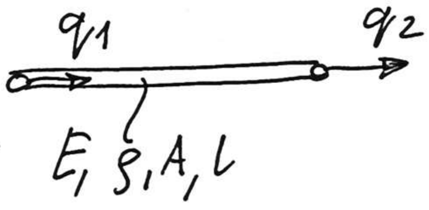
$$\int_0^{l_e} N_1 N_1 \, d\xi = \int_0^{l_e} \left(1 - \frac{\xi}{l_e}\right)^2 d\xi = \int_0^{l_e}\left(1 - \frac{2\xi}{l_e} + \frac{\xi^2}{l_e^2}\right) d\xi = l_e - \frac{l_e^2}{l_e} + \frac{l_e^3}{3 l_e^2} = \frac{l_e}{3}$$

$$\underset{2\times 2}{[m]_e} = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad [m_L]_e = \frac{\rho A l_e}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE. FIND NATURAL ANGULAR FREQUENCIES AND VIBRATION MODES FOR A SINGLE BAR ELEMENT.

CONSIDER UNCONSTRAINED AND CONSTRAINED BAR.

UNCONSTRAINED BAR:



$$\underset{2\times 2}{[k]_e} = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \underset{2\times 2}{[K]}$$

$$\underset{2\times 2}{[m]_e} = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \underset{2\times 2}{[M]}$$

$$\Big(\underset{2\times 2}{[K]} - \omega^2 \underset{2\times 2}{[M]}\Big)\underset{2\times 1}{\{q\}} = \underset{2\times 1}{\{0\}}$$

$$\det\Big(\underset{2\times 2}{[K]} - \omega^2 \underset{2\times 2}{[M]}\Big) = 0$$

$$\det\left(\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \cdot \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 0$$

$$\frac{EA}{l}\cdot \det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\omega^2 \rho l^2}{6E}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 0$$

$$\lambda = \frac{\omega^2 \rho l^2}{6E} \Rightarrow \det\begin{bmatrix} 1-2\lambda & -(1+\lambda) \\ -(1+\lambda) & 1-2\lambda \end{bmatrix} = 0$$

$$(1-2\lambda)^2-(1+\lambda)^2=0$$
$$1-4\lambda+4\lambda^2-1-2\lambda-\lambda^2=0$$
$$3\lambda^2-6\lambda=0$$
$$3\lambda(\lambda-2)=0$$
$$\Downarrow \qquad \searrow$$
$$\lambda_1=0 \qquad \lambda_2=2$$

$$\omega_i=\sqrt{\frac{6E\lambda_i}{\rho l^2}}$$

$$\omega_1=\sqrt{\frac{6E\lambda_1}{\rho l^2}}=0$$

$$\omega_2=\sqrt{\frac{6E\lambda_2}{\rho l^2}}=\frac{2\sqrt{3}}{l}\sqrt{\frac{E}{\rho}}$$

VIBRATION MODES

$$\frac{EA}{l}\left(\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}-\lambda\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\cdot\{q\}=\{0\}$$

$$\begin{bmatrix}1-2\lambda & -(1+\lambda)\\ -(1+\lambda) & 1-2\lambda\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

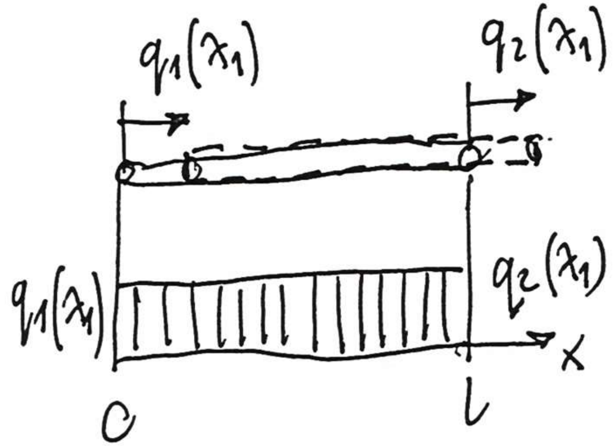
$\lambda_1 = 0$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1(x_1) \\ q_2(x_1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left.\begin{aligned} 1 \cdot q_1(x_1) - 1 \cdot q_2(x_1) &= 0 \\ -1 \cdot q_1(x_1) + 1 \cdot q_2(x_1) &= 0 \end{aligned}\right\} \text{ equations are linearly dependent}$$

$$q_1(x_1) = q_2(x_1)$$

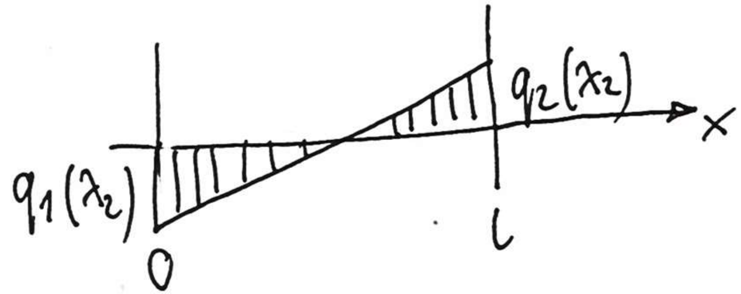
MOVEMENT AS A RIGID BODY



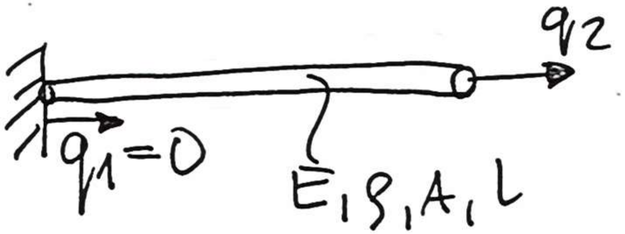
$\lambda_2 = 2$:

$$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \cdot \begin{Bmatrix} q_1(\lambda_2) \\ q_2(\lambda_2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$-3\,q_1(\lambda_2) - 3\,q_2(\lambda_2) = 0 \quad \Rightarrow \quad q_2(\lambda_2) = -q_1(\lambda_1)$$



CONSTRAINED BAR :



$$\left(\begin{bmatrix} & \\ & \end{bmatrix} - \omega^2 \begin{bmatrix} & \\ & \end{bmatrix}\right)\begin{Bmatrix}0\\ q_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

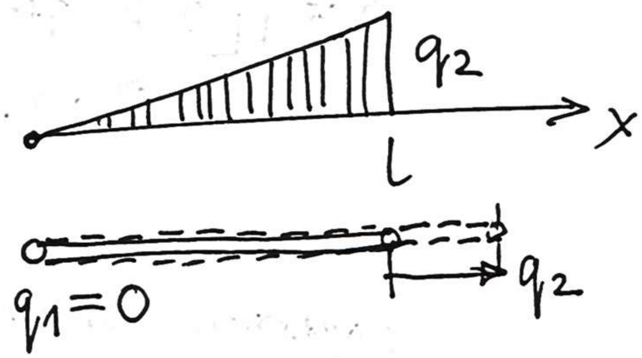
$$\left(\frac{EA}{L} - \omega_1^2\,\frac{\rho A L}{3}\right)\cdot q_2 = 0$$

$$\omega_1 = \frac{\sqrt{3}}{L}\sqrt{\frac{E}{\rho}}$$

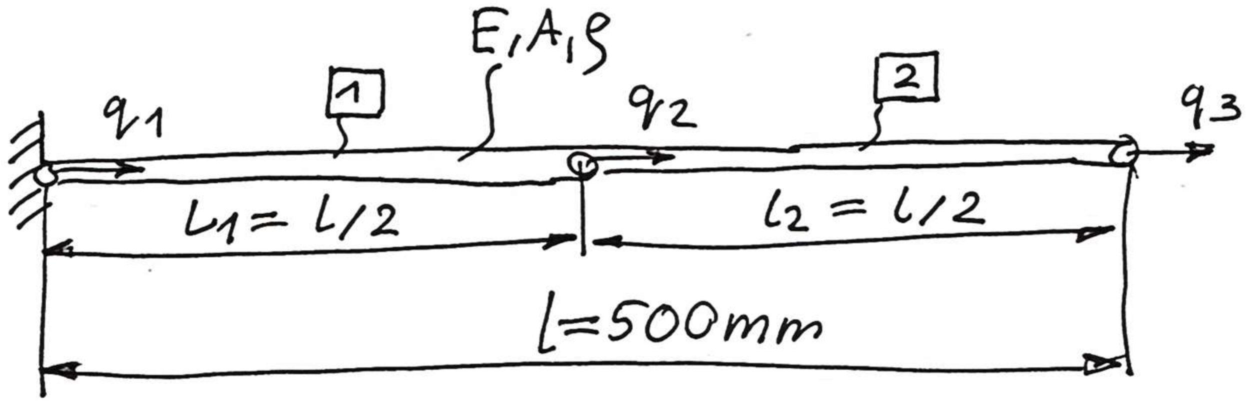
Analytical solution : $\bar{\omega}_i = \frac{(i-0.5)\pi}{l}\sqrt{\frac{E}{\rho}}$

$$\bar{\omega}_1 = \frac{1.5708}{l}\sqrt{\frac{E}{\rho}}$$

Relative error: $\Delta\omega_1 = \frac{\omega_1 - \bar{\omega}_1}{\bar{\omega}_1} = 10\,\%$



EXAMPLE. FIND NATURAL FREQUENCIES AND VIBRATION MODES FOR A BAR CONSTRAINED AT ONE END USE 2 FES.



$$E = 2\cdot 10^5 \ MPa$$

$$\rho = 7.8\cdot 10^3 \frac{kg}{m^3} = 7.8\cdot 10^3 \ \frac{kg\, m}{s^2 m^3}\cdot \frac{s^2}{m} =$$

$$= 7.8\cdot 10^3 \ \frac{N s^2}{m^4} = 7.8\cdot 10^{-9} \ \frac{N s^2}{mm^4}$$

STEEL

$$A = 100 mm^2 \ , \ l_1 = l_2 = l/2$$

FE solution:

$$\underset{1\times 3}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3 \rfloor$$

$$[k]_1 = \frac{EA}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = [k]_2 = \frac{2EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[m]_1 = [m]_2 = \frac{\rho A l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A l}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\underset{3\times 3}{[K]} = \begin{bmatrix} [k]_1 & 0 \\ 0 & [k]_2 \end{bmatrix} = \frac{2EA}{l}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underset{3\times 3}{[M]} = \begin{bmatrix} [m]_1 & \\ & [m]_2 \end{bmatrix} = \frac{\rho A l}{12}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

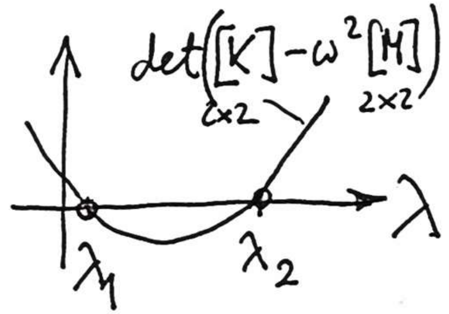
$$\left(\underset{3\times3}{[K]} - \omega^2 \underset{3\times3}{[M]}\right)\cdot \underset{3\times1}{\{q\}} = \underset{3\times1}{\{0\}} \qquad , \qquad q_1 = 0$$

$$\left(\frac{2EA}{l}\cdot\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \cdot \frac{\rho A l}{12}\cdot\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)\cdot\begin{Bmatrix} 0 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{2EA}{l}\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \underbrace{\frac{\omega^2 \rho l^2}{24E}}_{\lambda}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\cdot\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det \begin{bmatrix} 2-4\lambda & -(1+\lambda) \\ -(1+\lambda) & 1-2\lambda \end{bmatrix} = 0$$



$$(2-4\lambda)(1-2\lambda) - \big(-(1+\lambda)\big)\cdot\big(-(1+\lambda)\big) = 0$$

$$2-4\lambda-4\lambda+8\lambda^2-1-2\lambda-\lambda^2=0$$

$$7\lambda^2-10\lambda+1=0 \;;\; \Delta=(-10)^2-4\cdot 1\cdot 7 = 72 \Rightarrow \sqrt{\Delta} = 6\sqrt{2}$$

$$\lambda_1 = \frac{-(-10)-6\sqrt{2}}{14} = 0.1082\,, \quad \lambda_2 = \frac{-(-10)+6\sqrt{2}}{14} = 1.3204$$

FIRST VIBRATION MODE :

$$\omega_1 = \sqrt{\frac{24E\lambda_1}{\rho l^2}} = \sqrt{\frac{24 \cdot 2 \cdot 10^5\,MPa \cdot 0.1082}{7.8 \cdot 10^{-9}\frac{Ns^2}{mm^4} \cdot (500mm)^2}} = 16320\ \frac{1}{s}$$

$$f_1 = \frac{\omega_1}{2\pi} = 2597.4\,Hz$$

Analytical solution: $\bar{\omega}_i = \frac{(i-0.5)\pi}{l}\sqrt{\frac{E}{\rho}}$, $\bar{\omega}_1 = \frac{\pi}{2l}\sqrt{\frac{E}{\rho}}$

Relative error : $\Delta\omega_1 = \frac{\omega_1 - \bar{\omega}_1}{\bar{\omega}_1} = 2.6\,\%$

SECOND VIBRATION MODE :

$$\omega_2 = \sqrt{\frac{24E\lambda_2}{\rho l^2}} = 57011\ \frac{1}{s} \quad , \quad f_2 = \frac{\omega_2}{2\pi} = 9073.5\,Hz$$

$$\bar{\omega}_2 = \frac{3\pi}{2l}\sqrt{\frac{E}{\rho}} \quad , \quad \Delta\omega_2 = \frac{\omega_2 - \bar{\omega}_2}{\bar{\omega}_2} = 19.5\,\%$$

EIGENVECTORS: $(i=1,2)$:

$$\left(\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix} - \lambda_i\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}q_2(\lambda_i)\\ q_3(\lambda_i)\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\begin{cases}(2-4\lambda_i)\cdot q_2(\lambda_i) - (1+\lambda_i)\cdot q_3(\lambda_i) = 0\\ -(1+\lambda_i)\cdot q_2(\lambda_i) + (1-2\lambda_i)\cdot q_3(\lambda_i) = 0\end{cases}$$

two linearly dependent equations

$$\rightarrow \quad q_2(\lambda_i) = \frac{1+\lambda_i}{2-4\lambda_i}\cdot q_3(\lambda_i)$$

NORMALIZATION $\lfloor q \rfloor_{1\times 3} \cdot [I]_{3\times 3} \cdot \{q\}_{3\times 1} = 1$

$$\lfloor 0,\ q_2(\lambda_i),\ q_3(\lambda_i) \rfloor \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ q_2(\lambda_i) \\ q_3(\lambda_i) \end{Bmatrix} = 1$$

$$\lfloor 0,\ q_2(\lambda_i),\ q_3(\lambda_i) \rfloor \cdot \begin{Bmatrix} 0 \\ q_2(\lambda_i) \\ q_3(\lambda_i) \end{Bmatrix} = 1$$

$$\left(q_2(\lambda_i)\right)^2 + \left(q_3(\lambda_i)\right)^2 = 1$$

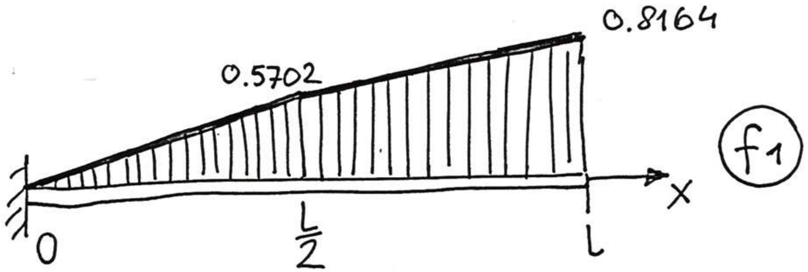
$$\frac{(1+\lambda_i)^2}{(2-4\lambda_i)^2} \cdot \left(q_3(\lambda_i)\right)^2 + \left(q_3(\lambda_i)\right)^2 = 1$$

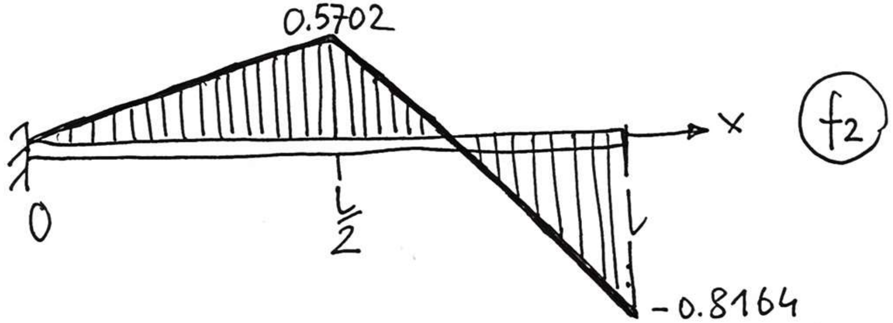
$$\frac{(1+\lambda_i)^2 + (2-4\lambda_i)^2}{(2-4\lambda_i)^2} \cdot \left(q_3(\lambda_i)\right)^2 = 1$$

$$q_3(\lambda_i) = \frac{2 - 4\lambda_i}{\sqrt{(1+\lambda_i)^2 + (2-4\lambda_i)^2}}$$

$$q_2(\lambda_i) = \frac{1 + \lambda_i}{\sqrt{(1+\lambda_i)^2 + (2-4\lambda_i)^2}}$$

i	λ_i	$q_2(\lambda_i)$	$q_3(\lambda_i)$	f_i [Hz]
1	0.1082	0.5702	0.8164	2597.4
2	1,3204	0.5702	−0.8164	9073.5





ANSYS:

2 FEs

10 FEs

	1st mode		*2nd mode*	
theory	**2531.854 Hz**	relative error	**7595.545 Hz**	relative error
2FEs	2597.294 Hz	2.6%	9074.278 Hz	19.5%
10FEs	2534.5 Hz	0.1%	7666 Hz	0.9%